are plotted in that figure. In this case all of these are straight line segments with the boundary drawn by the dash line parallel to the singular straight line.

In concluding the authors thank V. A. Panina and L. P. Frolova for their assistance, and A. G. Kulikovskii for useful discussions.

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UDC 532.526

ON THE BOUNDARY LAYER ON A PARTLY MOBILE SURFACE

PMM Vol. 40, № 3, 1976, pp. 479-489

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(Received March 31, 1975)

Transition of an incompressible boundary layer from the stationary section of a streamlined surface to its mobile section is considered under conditions of stabilized flow. Owing to the motion of a part of the surface a discontinuity of boundary conditions occurs at the surface. It is assumed that the presence of singularity in the boundary conditions does not affect the first approximation boundary layer upstream of the discontinuity line. The problem thus stated was first considered by Mager [1], who obtained an approximate solution for the simplest case of flow past a plate with the unperturbed stream in the form of a Blasius flow and the aft section of the plate moving perpendicularly to the basic stream.

The aim of this paper is the derivation of a solution of equations of the boundary layer in the neighborhood of the discontinuity line on the mobile section in the general case of three-dimensional flows. The solution upstream of that line is assumed known. The method used here may be considered as a generalization of the method of continuation [2] to the case of the three-dimensional boundary layer. A similar scheme of solution derivation for two-dimensional problems of a compressible boundary layer was proposed in [3].

1. Basic assumptions. The laminar flow of a viscous incompressible fluid past some surface is considered in the approximation of the boundary layer theory. The surface consists of a mobile and a fixed section which are separated by a line along which an abrupt change of boundary conditions takes place. Upstream of the separation line lies a region of unperturbed flow, in this case the boundary layer on the fixed part of the surface (it is assumed to be free of separation).

We make the following assumptions.

1°. The shape and motion of the rear segment are such that in a fixed reference system the surface as a whole can be specified by the parametric equation $\mathbf{r} = \mathbf{r}(x, y)$, where $\mathbf{r}$ is the radius vector of a point of the surface, which is independent of time. The sets of curves $x = \text{const}$ and $y = \text{const}$ define the fixed system of curvilinear coordinates on the considered surface. In the case of an orthogonal coordinate system the time independent parametrization of the surface makes it possible to retain the conventional form of equations of a steady boundary layer [4]. In dimensionless form we have

$$(h_2 u)_x + (h_1 v)_y + (h_1 h_2 w)_z = 0 \qquad (1.1)$$
$$h_2 u u_x + h_1 v u_y + h_1 h_2 w u_z + v(u h_{1y} - v h_{2x}) = -h_2 p_x + h_1 h_2 u_{zz}$$
$$h_2 u v_x + h_1 v v_y + h_1 h_2 w v_z + u(v h_{2x} - u h_{1y}) = -h_1 p_y + h_1 h_2 v_{zz}$$
$$h_1 = |\mathbf{r}_x|, \quad h_2 = |\mathbf{r}_y|, \quad R = UL/\nu$$

where $zR^{-1/2}$ is the distance measured along a normal to the surface; u, v and $wR^{-1/2}$ are velocity components in directions x, y and z, respectively; p is the pressure defined by the function of coordinates x and y known from the solution of the problem of inviscid flow past bodies, and R is the Reynolds number. All linear dimensions, velocity components and pressure are made dimensionless by relating them to some characteristic length L, characteristic velocity U, and to ρU^2, respectively.

2°. The rear segment is a rigid surface resistant to deformation whose motion is plane-parallel and the trajectories of its points do not intersect the discontinuity line. The velocity distribution in coordinates x, y are assumed to be independent of time. Hence with allowance for Assumption 1° it follows that the rear segment is either a cylindrical surface of infinite span or a surface of revolution. In the first case we have a translational motion along a generatrix, and in the second a uniform rotation about the axis of symmetry.

3°. It is possible to select, at least in the neighborhood of the discontinuity line, an orthogonal system of coordinates x, y in such a way that trajectories of points of the mobile segment belong to the set $x = \text{const}$. By virtue of 2° it is possible to assume that $x = 0$ at the discontinuity line and the unperturbed flow is situated in the region $x < 0$. In such system of coordinates the boundary conditions for Eqs. (1.1) are of the form

$$u(x, y, 0) = v(x, y, 0) = w(x, y, 0) = 0 \quad \text{for} \quad x < 0 \qquad (1.2)$$
$$u(x, y, 0) = w(x, y, 0) = 0; \; v(x, y, 0) = v_b(x) \quad \text{for} \quad x > 0 \qquad (1.3)$$
$$\lim_{z\to\infty} u(x, y, z) = u_e(x, y), \quad \lim_{z\to\infty} v(x, y, z) = v_e(x, y) \qquad (1.4)$$

where v_b is the velocity of motion of the surface and u_e and v_e are velocity components of the external inviscid flow. It is assumed that in the region of the discontinuity line functions h_1, h_2, p and v_b can be represented by power series (superscripts minus and plus relate, respectively, to quantities $x < 0$ and $x > 0$)

$$h_1(x, y) = \sum_{i=0}^{\infty} l_i^{\mp}(y) x^i, \quad h_2(x, y) = \sum_{i=0}^{\infty} m_i^{\mp}(y) x^i \tag{1.5}$$

$$p(x, y) = \sum_{i=0}^{\infty} p_i^{\mp}(y) x^i; \quad l_0^+ = l_0^- = l_0; \quad m_j^+ = m_j^- = m_j,$$

$$p_j^+ = p_j^- = p_j, \quad j = 0, 1$$

$$v_b(x) = \sum_{i=0}^{\infty} v_{bi} x^i \tag{1.6}$$

Owing to the chosen system of coordinates l_i^+ and v_{bi} are independent of y.

We consider the case when every point of line $x = 0$ is intersected by a surface streamline of the unperturbed flow. For $x = 0$ the velocity profiles are assumed to be known

$$u(0, y, z) = u_0(y, z), \quad v(0, y, z) = v_0(y, z) \tag{1.7}$$

Profiles (1.7) are taken as the input data for Eqs. (1.1) for calculating the boundary layer downstream of the discontinuity line. For small z these can be represented by the series

$$u_0(y, z) = \sum_{i=1}^{\infty} \frac{a_i(y)}{i!}, \quad v_0(y, z) = \sum_{i=1}^{\infty} \frac{b_i(y)}{i!} \quad (a_1 > 0) \tag{1.8}$$

The following conditions of matching profiles (1.7) with boundary conditions (1.2):

$$l_0 a_2 = p_1, \quad m_0 b_2 = p_{0y}, \quad a_3 = 0, \ \ldots \tag{1.9}$$

can be derived from Eqs. (1.1).

In the boundary layer on the mobile section of the surface in the vicinity of the discontinuity line it is possible to distinguish two regions: the inner sublayer which, owing to viscosity force, is affected by changing boundary conditions and the outer region where the change of boundary conditions affects the flow by way of interaction with the viscous sublayer. The extended variable

$$\mu = k(y) \frac{z}{\delta(x)}, \quad k(y) = \left(\frac{a_1}{3l_0}\right)^{1/3} > 0, \quad \delta(x) = x^m \quad (0 < m < 1) \tag{1.10}$$

is introduced in the inner region. The meaning of constant m will be explained below. Solution of Eqs. (1.1) with boundary conditions (1.3) and (1.4), and initial conditions (1.7) is sought in the form of matching the series formulated in the outer region for $x \to 0$ and fixed y and z, and in the inner region for fixed y and μ.

2. The inner problem. The principal terms of external expansions for velocity components u and v are, respectively, u_0 and v_0 whose behavior at small z is specified by formulas (1.8). Matching can be ensured by specifying inner expansions of functions u and v for form

$$u \sim \delta \sum_{n=0}^{\infty} \delta^n k^{2-n} F_n(y, \mu), \quad v \sim \sum_{n=0}^{\infty} \delta^n k^{-n} G_n(y, \mu) \tag{2.1}$$

As the corollary of (1.8) and (1.10) conditions

$$\lim_{\mu \to \infty} \frac{F_n(y, \mu)}{\mu^{n+1}} = \frac{3l_0 a_{n+1}}{(n+1)!\, a_1}, \quad \lim_{\mu \to \infty} \frac{G_n(y, \mu)}{\mu^n} = \frac{b_n}{n!} \quad (b_0 \equiv 0) \tag{2.2}$$

must be satisfied for considerable μ.

We represent component w by the series

$$w \sim \frac{\delta_1(x)}{3l_0} \sum_{n=0}^{\infty} \delta^n k^{1-n} H_n(y, \mu) \qquad (2.3)$$

If function $\delta(x)$ and $\delta_1(x)$ are determined, the substitution of expansions (2.1) and (2.3) together with (1.5) and (1.6) into Eqs. (1.1) and boundary conditions (1.3) yields a sequence of problems for the determination of functions F_n, G_n and H_n $(n = 0, 1, 2, \ldots)$. To take into account in the first approximation equations (relative to F_0, G_0 and H_0) the inertia, and also the viscous term, we set

$$m = {}^1/_3, \quad \delta_1(x) = \delta^{-1}(x) \qquad (2.4)$$

Then in the zero approximation the problem reduces to solving the system of equations with boundary conditions

$$F_0 - \mu F_0' + H_0' = 0, \quad 3l_0 F_0'' + (\mu F_0' - H_0) F_0' - F_0^2 = 0$$

$$3l_0 G_0'' + (\mu F_0 - H_0) G_0' = 0$$

$$F_0(y, 0) = H_0(y, 0) = 0, \quad G_0(y, 0) = v_{b0}, \quad \lim_{\mu\to\infty} \frac{F_0(y, \mu)}{\mu} = 3l_0$$

$$\lim_{\mu\to\infty} G_0(y, \mu) = 0$$

where the prime denotes differentiation with respect to μ. The solution of this problem is

$$F_0 = 3l_0\mu, \quad G_0 = v_{b0}\left(1 - c\int_0^{\mu} \exp\left(-\frac{t^3}{3}\right) dt\right), \quad H_0 \equiv 0 \qquad (2.5)$$

$$c^{-1} = \int_0^{\infty} \exp\left(-\frac{t^3}{3}\right) dt = \frac{\Gamma(^1/_3)}{\sqrt[3]{9}} \quad (c \approx 0.776458)$$

Using (2.5) for the n-th $(n > 0)$ approximation we obtain equations of the form

$$\psi_n''' + \mu^2\psi_n'' - (n+2)\mu\psi_n' + (n+2)\psi_n = Q_n \qquad (2.6)$$

$$G_n'' + \mu^2 G_n' - n\mu G_n = R_n - \frac{n+2}{3l_0} G_0'\psi_n \qquad (2.7)$$

$$H_n = \mu\psi_n' - (n+2)\psi_n + S_n \qquad (2.8)$$

Function ψ_n is linked to F_n by the relationship

$$\psi_n(y, \mu) = \int_0^{\mu} F_n(y, t)\, dt \qquad (2.9)$$

Functions Q_n, R_n and S_n depend on parameters of preceding approximations from the zero one to the $(n-1)$-st inclusive. General formulas for Q_n, R_n and S_n are not given here owing to their unwieldiness, although their derivation is not particularly difficult. It should be noted that Eqs. (2.6) differ from those obtained in [2], when solving plane problems of continuation, only by their right-hand sides.

Boundary conditions for Eqs. (2.6) and (2.7) with $\mu \to \infty$ are specified by formulas (2.2). Furthermore, in accordance with (1.3), (1.6) and (2.9)

$$\psi_n(y, 0) = \psi_n'(y, 0) = 0, \quad G_n(y, 0) = \begin{cases} k^n v_{bi}, & n = 3i \\ 0, & n \neq 3i \end{cases} \quad (i = 1, 2, 3 \ldots) \qquad (2.10)$$

All inner problems can be solved owing to their uniformity in the same way as the first and second approximation problems are solved below. It should be noted that the volume of computation of higher approximations sharply increases.

Using the first of conditions (1.9), for the first approximation problem we can obtain

$$Q_1 = \frac{3l_0a_2}{a_1} - \frac{3m_1v_{b0}^2}{m_0a_1}\varphi_0'^2, \quad R_1 \equiv 0, \quad \varphi_0(\mu) = \frac{1}{v_{b0}}\int_0^\mu G_0(t)\,dt \qquad (2.11)$$

The behavior of any solution of the first approximation equations that for $\mu \to \infty$ satisfies condition (2.2) is defined by

$$\psi_1 = \frac{3l_0a_2}{a_1}\frac{\mu^3}{6} + A_1\mu + o(\mu^{-N}), \quad G_1 = b_1\mu + o(\mu^{-N}) \qquad (2.12)$$

Coefficient A_1 (y) is not determined by the condition at infinity, and $o(\mu^{-N})$ denotes a supplement that decreases faster than any negative power of μ.

In conformity with formulas (2.11) and (2.12), we represent ψ_1 (y, μ) as

$$\psi_1(y, \mu) = \frac{3l_0a_2}{a_1}\psi_{10}(\mu) - \frac{3m_1v_{b0}^2}{m_0a_1}\psi_{11}(\mu) \qquad (2.13)$$

Functions ψ_{10} and ψ_{11} satisfy the following equations and boundary conditions:

$$\psi_{10}''' + \mu^2\psi_{10}'' - 3\mu\psi_{10}' + 3\psi_{10} = 1; \quad \psi_{10}(0) = \psi_{10}'(0) = 0 \qquad (2.14)$$
$$\lim_{\mu\to\infty}\frac{\psi_{10}(\mu)}{\mu^3} = \frac{1}{6}$$

$$\psi_{11}''' + \mu^2\psi_{11}'' - 3\mu\psi_{11}' + 3\psi_{11} = \varphi_0'^2; \qquad (2.15)$$
$$\psi_{11}(0) = \psi_{11}'(0) = 0, \; \lim_{\mu\to\infty}\frac{\psi_{11}(\mu)}{\mu^3} = 0$$

The solution of problem (2.14) is

$$\psi_{10} = \frac{\mu^3}{6} \qquad (2.16)$$

and that of problem (2.15) is derived numerically.

For considerable μ

$$\psi_{11} = A_{11}\mu + o(\mu^{-N}) \qquad (2.17)$$

Since the constant A_{11} is determined by the specified boundary conditions at zero and is not a priori known, it is convenient to use the auxilliary solution χ (μ) of Eq. (2.15) which satisfies conditions

$$\chi(0) = 0; \quad \chi(\mu) = \mu + o(\mu^{-N}), \quad \mu \to \infty \qquad (2.18)$$

Functions ψ_{11} and χ are linked by the relationship

$$\psi_{11}(\mu) = \chi(\mu) - \chi'(0)\mu, \quad A_{11} = 1 - \chi'(0)$$

The numerical determination of χ (μ) can be achieved by passing to the boundary value problem of Eq. (2.15) at its end segment of integration, chosen so that at its right-hand end it is possible to specify with reasonable accuracy function χ and its first derivative by the asymptotic formula (2.18) in which the terms o (μ^{-N}) are rejected.

Preliminary estimates and trial computations had shown that for this and all subsequent problems of numerical integration it is sufficient to use [0, 6] as such segment. Function $\psi_{11}' \times 10$ is shown in Fig. 1 (curve 1) with $A_{11} = -0.081838$. It follows from (2.13), (2.16) and (2.17) that in (2.12)

$$A_1 = -\frac{3m_1 v_{b0}^2}{m_0 a_1} A_{11} \tag{2.19}$$

For $n = 1$ we use (2.11) and (2.13) for transforming the right-hand side of Eq. (2.7) to

$$R_1 - \frac{1}{l_0} G_0' \psi_1 = -\frac{3a_2 v_{b0}}{a_1} \varphi_0'' \psi_{10} + \frac{3m_1 v_{b0}^3}{l_0 m_0 a_1} \varphi_0'' \psi_{11} \tag{2.20}$$

We represent the solution of the considered equation in the form

$$G_1(y, \mu) = b_1 G_{10}(\mu) - \frac{3a_2 v_{b0}}{a_1} G_{11}(\mu) + \frac{3m_1 v_{b0}^3}{l_0 m_0 a_1} G_{12}(\mu) \tag{2.21}$$

where functions G_{10}, G_{11} and G_{12} satisfy, with allowance for (2.20), equations with

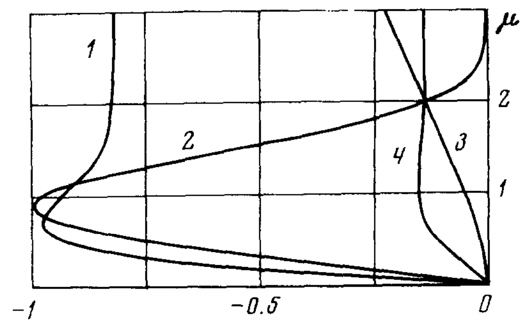

Fig. 1

boundary conditions

$$G_{10}'' + \mu^2 G_{10}' - \mu G_{10} = 0; \quad G_{10}(0) = 0, \quad \lim_{\mu\to\infty} \frac{G_{10}(\mu)}{\mu} = 1 \tag{2.22}$$

$$G_{1i}'' + \mu^2 G_{1i}' - \mu G_{1i} = \varphi_0'' \psi_{1, i-1}; \quad G_{1i}(0) = 0, \quad \lim_{\mu\to\infty} \frac{G_{1i}(\mu)}{\mu} = 0 \tag{2.23}$$

$(i = 1, 2)$

that correspond to (2.10) and (2.2).

The solution of problem (2.22) is

$$G_{10} = \mu \tag{2.24}$$

Since ψ_{10} and ψ_{11} have been already determined, G_{11} is obtained from (2.23) in simple quadratures

$$G_{11} = \mu \left[\frac{1}{30} - c \int_0^{\mu} \left(\frac{t^3}{6} - \frac{t^6}{30} \right) \exp\left(-\frac{t^3}{3} \right) dt \right] + \frac{c}{30} \mu^5 \exp\left(-\frac{\mu^3}{3} \right)$$

and G_{12} is determined by integrating Eq. (2.25) over the segment [0, 6], with the condition at $\mu = 6$, where G_{12} virtually vanishes, substituted for conditions at infinity. Function $G_{12} \times 10^2$ is shown in Fig. 1 (curve 2).

Using (2.13), (2.16), (2.21) and (2.24), for the second approximation we obtain

$$Q_2 = -\frac{6m_1b_1v_{b0}}{m_0a_1}\mu\varphi_0' + \frac{l_0v_{b0}}{m_0a_1}\frac{da_1}{dy}(2\mu\varphi_0' + \varphi_0) + \tag{2.25}$$
$$\frac{3m_1^2v_{b0}^4}{l_0m_0^2a_1^2}(2\psi_{11}'^2 - 3\psi_{11}\psi_{11}'' - 6\varphi_0'G_{12}) -$$
$$\frac{3m_1a_2v_{b0}^2}{m_0a_1^2}\left(2\mu^2\psi_{11}' - 3\mu\psi_{11} - \frac{\mu^3}{2}\psi_{11}'' - 6\varphi_0'G_{11}\right)$$

The asymptotic behavior of any solution of Eq.(2.6) for considerable μ and $n = 2$ with the right-hand side of (2.25) which satisfies condition (2.2) (where by virtue of (1.9) $a_3 = 0$) is defined as follows:

$$\psi_2(y, \mu) = -\frac{3m_1a_2v_{b0}^2}{m_0a_1^2}\frac{A_{11}\mu^2}{2} + A_2\mu + \frac{l_0v_{b0}}{m_0a_1}\frac{da_1}{dy}\frac{L_0}{4} + \tag{2.26}$$
$$\frac{3m_1^2v_{b0}^4}{l_0m_0^2a_1^2}\frac{A_{11}^2}{2} + o(\mu^{-N}), \quad L_0 = \lim_{\mu\to\infty}\varphi_0(\mu) \approx 0.729011$$

The coefficient $A_2(y)$ is not determined by the condition for $\mu \to \infty$. We represent function $\psi_2(y, \mu)$ in the form

$$\psi_2(y, \mu) = -\frac{6m_1b_1v_{b0}}{m_0a_1}\psi_{21}(\mu) + \frac{l_0v_{b1}}{m_0a_1}\frac{da_0}{dy}\psi_{22}(\mu) - \tag{2.27}$$
$$\frac{3m_1a_2v_{b0}^2}{m_0a_1^2}\psi_{23}(\mu) + \frac{3m_1^2v_{b0}^4}{l_0m_0^2a_1^2}\psi_{24}(\mu)$$

By analogy with the first approximation $\psi_{2i}(\mu)$ satisfies equations of the kind of (2.6) for $n = 2$ and similar conditions at zero: $\psi_{2i}(0) = \psi_{2i}'(0) = 0$ $(i = 1, 2, 3, 4)$. The corresponding terms of (2.25) determine the right-hand sides of these equations, while the related terms of (2.26) define the solution behavior for considerable μ. Then

$$A_2(y) = -\frac{6m_1b_1v_{b0}}{m_0a_1}A_{21} + \frac{l_0v_{b0}}{m_0a_1}\frac{da_1}{dy}A_{22} - \tag{2.28}$$
$$\frac{3m_1a_2v_{b0}^2}{m_0a_1^2}A_{23} + \frac{3m_1^2v_{b0}^4}{l_0m_0^2a_1^2}A_{24}$$

where A_{2i} are constant coefficients at terms of order μ in asymptotic formulas for ψ_{2i} $(i = 1, 2, 3, 4)$ for $\mu \to \infty$.

For functions ψ_{21} and ψ_{22} we have the simple quadratic formulas

$$\psi_{21} = \frac{\mu}{8}\left[\left(\frac{\mu^3}{3} + 1\right)\varphi_0' + c\int_0^\mu \frac{t^3}{3}\exp\left(-\frac{t^3}{3}\right)dt - 1\right]$$

$$\psi_{22} = \frac{1}{4}(\varphi_0 - \mu) + 3\psi_{21}$$

Functions ψ_{23} and ψ_{24} were numerically determined by the same method as ψ_{11}. Functions ψ_{23}' (curve 3) and $\psi_{24}' \times 10$ (curve 4) are shown in Fig. 1; $A_{21} = -1/12$, $A_{22} = -1/2$, $A_{23} = 0.024772$ and $A_{24} = -0.014183$.

In Eq.(2.7) for $n = 2$ with allowance for (1.9) we have

$$R_2 = b_2 + \frac{v_{b0}^2}{3m_0a_1}\frac{da_1}{dy}\varphi_0\varphi_0'' + \frac{1}{3l_0}(G_1\psi_1' - 3\psi_1G_1') \tag{2.29}$$

It is possible to obtain from (2, 27) and (2, 29) the right-hand sides of that equation whose form determines, as in previous cases, the form of solution

$$G_2(y,\mu) = b_2 G_{20}(\mu) + \frac{v_{b0}^2}{3m_0 a_1}\frac{da_1}{dy}G_{21}(\mu) + \frac{m_1 b_1 v_{b0}^2}{l_0 m_0 a_1}G_{22}(\mu) - \qquad (2.30)$$

$$\frac{a_2^2 v_{b0}}{a_1^2}G_{23}(\mu) + \frac{m_1 a_2 v_{b0}^3}{l_0 m_0 a_1^2}G_{24}(\mu) - \frac{m_1^2 v_{b0}^5}{l_0^2 m_0^2 a_1^2}G_{25}(\mu)$$

Function G_{2i} satisfies equations of the kind of (2, 7) for $n = 2$ with corresponding right-hand sides and boundary conditions at zero $G_{2i}(0) = 0$ $(i = 0, 1 \ldots 5)$. The conditions for $\mu \to \infty$ can be obtained from the asymptotic formula

$$G_2(y,\mu) = b_2\frac{\mu^2}{2} - \frac{m_1 b_1 v_{b0}^2}{l_0 m_0 a_1}A_{11} + o(\mu^{-N}) \qquad (2.31)$$

The solution of the problem for G_{20} is $G_{20} = \mu^2/2$.

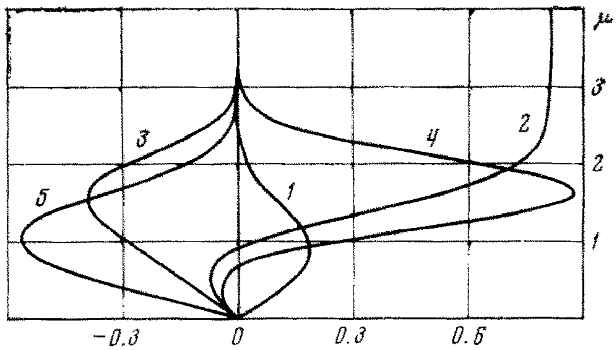

Fig. 2

The results of numerical integration of remaining functions are shown in Fig. 2 (curves 1—5 relate to functions G_{21}, $G_{22} \times 10$, $G_{23} \times 10$, $G_{24} \times 10^2$ and $G_{25} \times 10^2$). respectively). The used here and in the foregoing method of numerical solution of boundary value problems is the method of firing [5]. Computations were carried out on a computer. Functions H_1 and H_2 are readily determined by (2, 8), where $S_1 = S_2 \equiv 0$.

Thus formulas (2. 13), (2. 21), (2. 27) and (2. 30) provide solutions of first and second approximation of inner problems in the form of linear combination of certain "universal" functions of the variable μ, which are independent of initial and boundary conditions, and also of the shape of the surface and of the method of its parametrization. The knowledge of these functions makes it possible to define with specific accuracy the flow in the viscous sublayer near the separation line for any specific problem of the kind considered here.

If $m_1 = da_1/dy \equiv 0$ then, in conformity with (2. 13) and (2. 27), the discontinuity of boundary conditions does not affect in the first and second approximations the velocity profile in the x-direction. Its effect is also absent in higher approximations if the rear segment is a cylindrical surface and the initial (1. 7) and boundary (1. 4) conditions are independent of y, since then the first two of Eqs. (1. 1) are resolved independently of the third.

The corollary of (2. 5), (2. 13) and (2. 16) implies that for small x the relationships

$$u_z(x, y, 0) = a_1 - \delta \frac{3m_1 k^2 v_{b0}^2}{m_0 a_1} \psi_{11}''(0) + o(\delta), \quad \psi_{11}''(0) = -0.375839$$

are satisfied on the mobile part of the surface.

It follows from this that close to the wall the stream is accelerated in the x-direction when the rear segment represents a widening surface of revolution $(m_1 > 0)$, and is decelerated when the surface of revolution is a contracting one $(m_1 < 0)$. This feature is a consequence of the effect of centrifugal forces on the particles of fluid which are drawn into rotation.

3. The external problem. The form of external expansions of functions u, v and w is determined by the behavior of coefficients of corresponding internal expansions for considerable μ. The asymptotic formulas (2.12), (2.26) and (2.31) make it possible to extablish that for small δ in the external region

$$u = u_0(y, z) + \delta^2 u_2(y, z) + \delta^3 u_3(y, z) + o(\delta^3) \qquad (3.1)$$

$$v = v_0(y, z) + \delta^2 v_2(y, z) + \delta^3 v_3(y, z) + o(\delta^3)$$

$$w = \frac{1}{\delta} w_2(y, z) + w_3(y, z) + o(1)$$

By substituting (3.1) into the input equations (1.1) and equating terms of like order with respect to δ, we obtain equations for determining functions u_2, v_2, w_2 and u_3, v_3, w_3

$$u_2 = c_2 u_{0z}, \quad v_2 = c_2 v_{0z}, \quad w_2 = -\frac{2c_2}{3l_0} u_0$$

$$u_3 = c_3 u_{0z} + \frac{1}{m_0 u_0}(l_0 m_0 u_{0zz} - l_0 v_0 u_{0y} + m_1 v_0^2 - m_0 p_1) - \frac{u_{0z}}{m_0}\int_0^z \theta(y, t)\, dt$$

$$v_3 = c_3 v_{0z} + \frac{1}{m_0 u_0}(l_0 m_0 v_{0zz} - l_0 v_0 v_{0y} - m_1 u_0 v_0 - l_0 p_{0y}) - \frac{v_{0z}}{m_0}\int_0^z \theta(y, t)\, dt$$

$$w_3 = -\frac{c_3}{l_0} u_0 + \frac{u_0}{l_0 m_0}\int_0^z \theta(y, t)\, dt$$

$$\theta(y, z) = \frac{1}{u_0^2}[l_0(v_0 u_{0y} - u_0 v_{0y}) - m_1(u_0^2 + v_0^2) + m_0 p_1 - l_0 m_0 u_{0zz}]$$

which are simply integrable. Here $c_2(y)$ and $c_3(y)$ are arbitrary functions that are determined in the course of joining external and internal expansions.

Although the denominator of function θ is for $z \to 0$ of order z^2, there is no singularity at zero, since owing to boundary conditions (1.2) and the conditions of merging (1.9) the numerator is of the same order of smallness.

To carry out the joining we pass in the external expansion (3.1) to internal variables. Using (1.8) and (1.9) we obtain

$$u \sim \delta\left(\frac{a_1}{k}\mu + \ldots\right) + \delta^2\left(\frac{a_2}{k^2}\frac{\mu^2}{2} + c_2 a_1 + \ldots\right) + \delta^3\left(\frac{a_2 c_2}{k}\mu + c_3 a_1 + \ldots\right) + \ldots \quad (3.2)$$

By comparing (3.2) with the asymptotic formulas (2.12) and (2.26) we can verify that joining is ensured when

$$c_2 = kA_1/a_1, \quad c_3 = A_2/a_1 \quad (3.3)$$

Coefficients A_1 and A_2 are specified by formulas (2.19) and (2.28). A simple test shows that when (3.3) is satisfied, the conditions of joining expansions of functions v and w are also satisfied with the considered accuracy.

The effect of the viscous sublayer on the flow in the external region is generally that of inducing an abrupt change of boundary layer thickness. A general equation was obtained in [6] which is satisfied by the effective displacement thickness $R^{-1/2}\Delta(x, y)$ in the general case of three-dimensional boundary layer. In the notation used here

$$[h_2 u_e(\Delta - \Delta_1)]_x + [h_1 v_e(\Delta - \Delta_2)]_y = 0 \quad (3.4)$$

$$\Delta_1(x, y) = \int_0^\infty \left(1 - \frac{u}{u_e}\right) dz, \quad \Delta_2(x, y) = \int_0^\infty \left(1 - \frac{v}{v_e}\right) dz$$

With the use of series (2.1) and (3.1) it is possible to expand for small x functions Δ_1 and Δ_2, hence it is convenient to seek the solution of Eq. (3.4) in the form of an asymptotic series. Restricting the latter to the first two terms we obtain

$$\Delta(x, y) = D_0(y) + \delta^2 D_2(y) + o(\delta^2), \quad D_2 = -c_2 = \frac{3m_1 k v_{b0}^2}{m_0 a_1^2} A_{11} \quad (3.5)$$

where $D_0(y)$ is the displacement thickness of the unperturbed boundary layer at the discontinuity line. Since $A_{11} < 0$, hence by virtue of (3.5) for $m_1 > 0$ the displacement thickness becomes smaller than the thickness of the unperturbed boundary layer. This is caused by centrifugal forces which, as previously indicated, accelerate the stream along x and by that contribute to a free from separation flow past the surface. For $m_1 < 0$ the effect of these forces is opposite, which results in an increase of the boundary layer thickness.

It should be noted that the coefficients in the external expansions of u_3, v_3 and w_3 contain terms whose form is independent of the solution in the inner region (such terms appear in subsequent terms at the interval δ^3). It is simultaneously possible to separate in the asymptotic formulas for functions F_n and G_n, commencing with $n = 3$ and $\mu \to \infty$, terms that are independent of boundary conditions at the surface of the body and differ from those determined by conditions (2.2). The appearance of such terms in asymptotic formulas is related to the effect of solution in the external region on the solution in the sublayer. They ensure the joining with the independent terms in external solutions indicated previously.

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Translated by J. J. D.

UDC 532.526

THREE-DIMENSIONAL BOUNDARY LAYER IN A PARTLY IONIZED MULTICOMPONENT GAS

PMM Vol. 40, № 3, 1976, pp. 490-500

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(Received September 5, 1975)

A system of equations is derived for the three-dimensional boundary layer in a partly ionized multicomponent gas with frozen reactions under conditions of quasi-inertness and absence of external electromagnetic fields and of energy transfer by radiation. An analytical computation method based on the use of successive approximations is investigated. Variation of transfer coefficients across the boundary layer is taken into account by approximating the values of these at the external boundary and at the surface of the body. First approximation values of surface friction and heat exchange coefficients are obtained for the locally self-similar cases. An example of computation of the flow of frozen air past a cone with spherically blunted nose at an angle of attack is presented.

1. Let us consider the three-dimensional motion of a partly ionized multicomponent gas. If external electromagnetic fields are absent and the thermal diffusion effect is disregarded, the system of equations for a three-dimensional frozen boundary layer can be written as follows:

$$\frac{\partial}{\partial \xi}\left(\rho \sqrt{\frac{g}{g_{11}}} u\right) + \frac{\partial}{\partial \eta}\left(\rho \sqrt{\frac{g}{g_{22}}} w\right) + \sqrt{g}\,\frac{\partial \rho v}{\partial \zeta} = 0$$

$$\frac{\rho u}{\sqrt{g_{11}}}\frac{\partial c_i}{\partial \xi} + \frac{\rho w}{\sqrt{g_{22}}}\frac{\partial c_i}{\partial \eta} + \rho v \frac{\partial c_i}{\partial \zeta} + \frac{\partial I_i}{\partial \zeta} = 0, \quad i = 1, \ldots, N$$

$$\frac{u}{\sqrt{g_{11}}}\frac{\partial u}{\partial \xi} + \frac{w}{\sqrt{g_{22}}}\frac{\partial u}{\partial \eta} + v\frac{\partial u}{\partial \zeta} + A_1 u^2 + A_2 w^2 + A_3 uw =$$
$$\frac{A_4}{\rho} + \frac{1}{\rho}\frac{\partial}{\partial \zeta}\left(\mu \frac{\partial u}{\partial \zeta}\right), \quad \frac{\partial p}{\partial \zeta} = 0$$

$$\frac{u}{\sqrt{g_{11}}}\frac{\partial w}{\partial \xi} + \frac{w}{\sqrt{g_{22}}}\frac{\partial w}{\partial \eta} + v\frac{\partial w}{\partial \zeta} + B_1 u^2 + B_2 w^2 + B_3 uw =$$
$$\frac{B_4}{\rho} + \frac{1}{\rho}\frac{\partial}{\partial \zeta}\left(\mu \frac{\partial w}{\partial \zeta}\right)$$

$$\frac{\rho u}{\sqrt{g_{11}}}\frac{\partial H}{\partial \xi} + \frac{\rho w}{\sqrt{g_{22}}}\frac{\partial H}{\partial \eta} + \rho v \frac{\partial H}{\partial \zeta} =$$